International Journal of Theoretical Physics, Vol. 29, No. 10, 1990

Spontaneous Compactification and Quantized Charges

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Received May 2, 1990

We modify the Weinberg formalism relating gauge coupling constants to averaged circumferences of compact spaces by divorcing it from its original Kaluza–Klein geometrical context. The resulting more versatile formalism is then used to prove a theorem giving a formula for a new kind of quantized charge for the case of spontaneous compactification, defined as the situation where the solutions of the Euler–Lagrange field equations are periodic in one or more of the parameters of the noncompact invariance group of the action. The theorem is applied to the example of periodic waves.

1. INTRODUCTION

Weinberg (1983) considered a $(4+N)$-dimensional space with coordinates which could be separated into the coordinates x^{μ} of ordinary space-time plus the coordinates y^{n} of a compact N-dimensional manifold M. He showed, in this Kaluza (1921)–Klein (1926) type theory, that the gauge coupling constants associated with the isometry group of the compact space are given by

$$g_e = \frac{2\pi\kappa}{N_e\langle s^2(e, y)\rangle^{1/2}} \tag{1}$$

where $\kappa^2 \equiv 16\pi G$ and G is the Newtonian gravitational constant. The denominator includes the rms circumferences of M along certain curves defined by Weinberg and the valuedness N_e of the representation used. This generalizes earlier work by Klein (1926) and by Souriau (1963) relating the electronic charge to the radius of the fifth dimension in the $U(1)$ case to include more general non-Abelian theories (DeWitt, 1964; Scherk and Schwarz, 1975; Cho and Freund, 1975; Cho, 1975; Cremmer and Scherk, 1976; Freund and Rubin, 1980; Witten, 1981; Cremmer, 1982; Duff, 1982;

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Salam and Strathdee, 1982; Duff and Toms, 1982) with more than one compact dimension. κ enters (1) because Weinberg considers a geometrical theory with an action which is a generalization of the usual Hilbert action of general relativity to $4+N$ dimensions. The compact space M has an arbitrary scale factor associated with it, so that (1) does not allow the explicit calculation of individual coupling constants but, ultimately, only their ratios.

For completeness we should mention that grand unified theories (Ross, 1985) also determine coupling constants in some sense. In the general case, the diagonal generators span an Abelian subgroup which is isomorphic to $[R(1)]^p \times [U(1)]^q$. For a compact gauge group we get only $[U(1)]^n$. This leads the "charges" associated with that compact group to have ratios which are always the ratio of integers.

We will show below that much of the Weinberg formalism relating coupling constants to the circumference of a compact space also holds if we abandon the geometrical Kaluza-Klein context. The Kaluza-Klein approach is appropriate if the gauge group is fundamental and therefore possibly related to the geometry of a higher-dimensional manifold. In the following, we will be particularly interested in a theory whose gauge group is noncompact but whose solutions are periodic in at least one of the parameters of the noncompact group. We refer to this situation as "spontaneous compactification" in analogy with spontaneous symmetry breaking. Our use of this term differs from the more usual usage where a noncompact manifold actually physically curls up into a compact one as in superstring theory or in the work of Chodos and Detweiler (1980). In our case the resulting effective compact group arising from the assumed periodicity has nothing to do with fundamental physics or higher-dimensional geometry, and κ therefore does not enter in. κ in fact is replaced by a constant with the dimensions of action. We will use the Weinberg formalism but without the Kaluza-Klein context to prove a theorem that we get a new kind of quantized coupling constant obeying a formula analogous to (1) in the case of spontaneous compactification.

As an application of our formalism, we will show that periodic waves are an effective compactification of $T(4)$ to $U(1)$. The four-momentum is the quantized charge. We also speculate on other applications.

2. RELATING COUPLING CONSTANTS TO GROUP CIRCUMFERENCES NONGEOMETRICALLY

We would now like to consider the Weinberg (1983) formalism relating gauge coupling constants to rms compact space circumferences, but taken out of the Kaluza-Klein geometrical context. This is relevant for what we will do below and is also important in its own right since Kaluza-Klein theory has severe chirality problems (Witten, 1981) if one tries to use it for the unification of physics. Weinberg considers the situation where all eigenvalue of $e^{\alpha}t_{\alpha}$, for an arbitrary fixed vector e^{α}, are integer multiples of the eigenvalues g_e of lowest nonzero absolute value. The t_{α} are the Hermitian generators of the isometry groups of M in a given representation which is N_e valued for the subgroup generated by $e^{\alpha}t_{\alpha}$. The vector e^{α} is normalized according to

$$\delta_{\alpha\beta}e^{\alpha}e^{\beta}=1 \tag{2}$$

Weinberg shows that g_e is the physical gauge coupling constant and is related to the circumference of M associated with a given e^{α} and starting point y_0 by

$$s(e, y_0)=\frac{2\pi}{g_e N_e}[\tilde{g}_{nm}(y_0)\xi^n_{\alpha}(y_0)\xi^m_{\beta}(y_0)e^{\alpha}e^{\beta}]^{1/2} \tag{3}$$

where $\tilde{g}_{nm}(y)$ is the metric of the compact space with Killing vectors $\zeta^n_{\alpha}(y)$. So far Weinberg has just used group theory. The Kaluza-Klein geometrical theory provides the normalization of the Killing vectors appearing in (3) by giving

$$\langle\xi^n_{\alpha}(y)\xi^m_{\beta}(y)\tilde{g}_{nm}(y)\rangle=\kappa^2\delta_{\alpha\beta} \tag{4}$$

where the brackets denote an average over the compact manifold M. If (3) is averaged over the starting points y_0 and (2) and (4) are used, Weinberg's result (1) follows.

We see from the above that we can keep much of the Weinberg formalism and yet completely divorce it from the Kaluza-Klein formalism by abandoning (4) and replacing it with

$$\langle\xi^n_{\alpha}(y)\xi^m_{\beta}(y)\tilde{g}_{nm}(y)\rangle=\eta^2\delta_{\alpha\beta} \tag{5}$$

where η is a constant unrelated to any geometrical consideration. We can always pick and normalize Killing vectors to satisfy (5). For us, η is an arbitrary free parameter. The compact space now has lost any geometrical meaning it once had. Averaging (3) over the starting points of the paths and using (5) and (2) now gives

$$g_e=\frac{2\pi\eta}{N_e\langle s^2(e, y)\rangle^{1/2}} \tag{6}$$

We still have quantized charges related to the circumference of the compact space. Since the compact space has an arbitrary scale factor in it anyway, using an arbitrary η rather than κ from Kaluza-Klein theory adds no real arbitrariness to the result. In practice, only ratios of coupling constants can be determined in either case. Equation (6) is more versatile than (1) since it is released from the limitations of Kaluza-Klein theory. We apply this result in the next section to the situation where the compact space arises not from the fundamental geometry, but from "spontaneous compactification."

3. SPONTANEOUS COMPACTIFICATION

We have the following theorem.

Theorem. Assume that the action of a field theory is invariant under a noncompact continuous Lie group G. We then have a conserved Noether (1918) current and conserved charges Q_A. Also assume that we restrict our attention to solutions (fields) of the Euler-Lagrange field equations which are exactly periodic in terms of at least one of the parameters ε of G, so that the fields return to their original values when $\varepsilon \to \varepsilon + P$. Let Q be the specific conserved Noether charge associated with invariance of the action under changes in ε. We then have

$$Q = A/P \tag{7}$$

where A is a constant with dimensions of action and Q is now a quantized as well as conserved charge.

Proof. We refer to this situation where the solutions (fields) of a gauge theory are exactly periodic in at least one parameter of a noncompact invariance group of the action as spontaneous compactification. In a given physical situation, such periodic fields may or may not exist, of course. We give an example later. Due to the assumed periodicity, as far as these fields are concerned, a subgroup of the noncompact invariance group behaves as though it were compact and in fact $U(1)$. This subgroup is associated with the parameter ε in the theorem. As far as these fields are concerned, the action itself will also be invariant under this compactified group. Thus, we satisfy the compactness requirements of the development in the preceding section and can apply that formalism to the present situation. The compact space $S(1)$ is isomorphic to $U(1)$ and can be taken to be the compact manifold in Weinberg's formalism. This compact manifold arises from the assumed periodicity and not fundamentally from the geometry. Thus, Kaluza-Klein theory should play no role, but the modified version of Weinberg's formalism developed in the preceding section will apply. We note that in (6), $N_e = 1$ since we have $S(1)$ and $\langle s^2(e, y)\rangle^{1/2}$ is simply the circumference of the $S(1)$ space. This circumference is *known* in the present case and is P since the fields return to their original values when $\varepsilon \to \varepsilon + P$. We are letting ε itself serve as the coordinate of the compact $S(1)$ space. g_e in (6), as in Weinberg's work, is a quantized coupling constant or charge. Since Q is the conserved Noether charge associated with invariance of the action under changes in ε, we can identify g_e in (6) as Q when applying (6) to this specific situation of spontaneous compactification. Inserting these various quantities into (6) then gives, for the periodic fields under consideration, that

$$Q = \frac{2\pi\eta}{P} \tag{8}$$

η is the normalization constant for the Killing vectors appearing in (5). If we identify the constant A in the theorem with $2\pi\eta$, we have proven the theorem with the exception of the statement that A or η has dimensions of action, which we now show.

The η in (5) and (6) in the preceding section is completely arbitrary. When we apply that formalism to the present situation of spontaneous compactification, however, η in (8) must have dimensions of action because of the way Q and P are defined and are related to one another. Let us now show this. Consider an action

$$S = \int \mathscr{L}(\phi^\alpha(x), \partial^\mu \phi^\alpha(x))\, d^4x \tag{9}$$

which is invariant under an internal symmetry transformation of the field variables [we follow Ross (1985)]

$$\phi^\alpha(x) \to \phi^\alpha(x) + F_k^\alpha(x)\, \delta\omega^k \tag{10}$$

where $\delta\omega^k$ are group parameters. The variation of the action can be written

$$\delta S = -\int \partial_\mu [J_k^\mu \delta\omega^k]\, d^4x \tag{11}$$

where the conserved Noether (1918) current is given by

$$J_k^\mu = \frac{-\partial\mathscr{L}}{\partial(\partial^\mu \phi^\alpha)} F_k^\alpha \tag{12}$$

and the associated conserved charge is

$$Q_k = \int J_k^0\, d^3x \tag{13}$$

Space-time symmetries can be treated similarly. Relating this to our work above gives our $Q \equiv Q_k$ and our group parameter $\varepsilon \equiv \delta\omega^k$. Also, our P has the same dimensions as ε, so that (11) and (13) immediately give

$$\delta S \sim QP \tag{14}$$

in units. But $2\pi\eta \equiv A$ has the dimensions of QP from (8), so that A in (7) has the dimensions of action. This completes our proof of the theorem. ■

Note that in the defining relation (5) for $\eta \equiv A/2\pi$, $\tilde{g}_{nm}$, the metric of our compact $S(1)$ space, has a scale factor which is determined, since P represents the circumference of the compact space. The ξ^n_α Killing vectors, however, are essentially any constant angle in the $S(1)$ space. Thus, there is nothing to fix their normalization and A or η is unspecified. We end up with a situation similar to the Kaluza-Klein work of Weinberg. Weinberg has a known $\eta \equiv \kappa$ from the Kaluza-Klein geometry but an unknown scale of the compact space. This scale may perhaps be fixed by quantum considerations (Candelas and Weinberg, 1984). In our work, on the other hand, we have a known scale or circumference of our $S(1)$ space from the known periodic dependence of the fields on the group parameter, but an unknown η. We have shown above that periodic dependence of the fields on a parameter of a noncompact symmetry group leads to a new kind of quantized charge, analogous to the charges of Weinberg for the compact case. We look at a specific example in the next section.

4. APPLICATION TO TRANSLATIONAL INVARIANCE AND PERIODIC WAVES

Consider an action which is Poincaré invariant and in particular invariant under the noncompact continuous four-dimensional translation group $T(4)$. As is well known, the associated conserved Noether charge is the four-momentum P_μ. Now consider solutions to the field equations which are periodic plane waves. The periodicity must be *exact.* Let them travel in the $+x$ direction for definiteness. The plane wave is then given by any function $f(\xi)$, where $\xi \equiv x - ct$ if we further assume that the fields are massless and travel at the speed of light. From the plane wave periodicity, we have $f(\xi) = f(\xi + \lambda)$. Consider a translation in the x direction, for example. These solutions to the field equations are periodic in one of the parameters of the original noncompact $T(4)$ symmetry group, namely that having to do with displacements in x. Our theory can then be applied. The conserved Noether charge Q corresponding to invariance under x translation is now P_x, the periodicity interval P in the theorem is λ, and (7) gives

$$P_x = \frac{A}{\lambda} \tag{15}$$

where A is a constant of dimensions of action and p_x is *quantized.* One can view A as a free parameter which one finds experimentally to be equal to Planck's constant, at least in this example. As mentioned above, there is nothing in this formalism which fixes the magnitude of A. Its dimensions are fixed as those of action, however. The result (15) is rather remarkable, especially since our theorem says that P_x must be quantized, yet we are doing classical physics. We get a quantized P_x because our periodic wave is an identification of the x spatial dimension modulo the wavelength, i.e., this spatial dimension essentially becomes a circle $S(1)$ as far as the wave is concerned and the translation group becomes compactified. Compact groups give quantized charges. For periodic plane waves moving in other directions, we can easily generalize the above result and write

$$P_\mu = \frac{A}{2\pi} k_\mu \tag{16}$$

where k_μ is the wave vector.

Other applications of this spontaneous compactification formalism arise whenever fields are exactly periodic in one or more parameters of a noncompact invariance group of the action. For example, physics is invariant under the noncompact Poincaré group and in particular under Lorentz boosts. The conserved quantities corresponding to both rotations and boosts are $\varepsilon^{\alpha\beta\gamma\delta} x_\gamma P_\delta$. These include angular momentum for the compact rotations and the initial values of $\mathbf{x}_0$ for the boosts, if we also use conservation of P_μ from translational invariance. If we could arrange a physical situation where the solutions to the field equations were periodic in the boost parameters, $\xi_i = \tanh^{-1} v_i/c$, then our theorem says that the conserved quantities above corresponding to the boosts would become quantized as in (7).

ACKNOWLEDGMENT

I am indebted to Norio Nakagawa for conversations concerning this paper.

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